

- Solving Systems of Linear DEs by Elimination

The DE $y' + 2y + 3x = -1$ can be written in terms of the differential operator D , where D represents a derivative. In this form, the DE becomes $Dy + 2y + 3x = -1$. This can also be written $(D + 2)y + 3x = -1$. Our work in this lesson makes use of this notation. For this lesson, x and y are functions of t , so

$$Dx = \frac{dx}{dt}, Dy = \frac{dy}{dt}, \text{ and } D^2x = \frac{d^2x}{dt^2}, \text{ etc.}$$

Solve the given system of differential equations by elimination.

Example: ① $(D+1)x + (D-1)y = 2$
 ② $3x + (D+2)y = -1$

We can eliminate x or y by applying appropriate differential operators and subtracting.

Elim x by multiplying 3·① and applying $-(D+1)$ to ②

$$3(D+1)x + 3(D-1)y = 6$$

$$-3(D+1)x - (D+1)(D+2)y = -(D+1)(-1)$$

$$[3(D-1) - (D+1)(D+2)]y = 6 - (D+1)(-1)$$

$$(3D-3-D^2-3D-2)y = 7$$

$$(D^2+5)y = -7 \Rightarrow y''+5y = -7$$

$$y_c = c_1 \cos \sqrt{5}t + c_2 \sin \sqrt{5}t$$

$$g(t) = -7 \Rightarrow y_p = A \text{ (constant) nonhomogeneous}$$

$$y'' = 0 \Rightarrow 5A = -7 \Rightarrow A = -\frac{7}{5}$$

$$y(t) = C_1 \cos \sqrt{5}t + C_2 \sin \sqrt{5}t - \frac{7}{5}$$

$$\textcircled{2}: x = -\frac{1}{3} - \frac{1}{3}(D+2)y$$

$$Dy = -\sqrt{5}C_1 \sin \sqrt{5}t + \sqrt{5}C_2 \cos \sqrt{5}t$$

$$x = -\frac{1}{3} + \frac{\sqrt{5}}{3}C_1 \sin \sqrt{5}t - \frac{\sqrt{5}}{3}C_2 \cos \sqrt{5}t - \frac{2}{3}C_1 \cos \sqrt{5}t - \frac{2}{3}C_2 \sin \sqrt{5}t + \frac{14}{15}$$

$$x(t) = \frac{9}{15} + \left(\frac{\sqrt{5}}{3}C_1 - \frac{2}{3}C_2\right) \sin \sqrt{5}t + \left(-\frac{2}{3}C_1 - \frac{\sqrt{5}}{3}C_2\right) \cos \sqrt{5}t$$

$$C_3 = -\frac{\sqrt{5}}{3}C_2 - \frac{2}{3}C_1 \quad C_4 = \frac{\sqrt{5}}{3}C_1 - \frac{2}{3}C_2$$

$$y(t) = C_1 \cos \sqrt{5}t + C_2 \sin \sqrt{5}t - \frac{7}{5}$$

We have $y(t) = C_1 \cos \sqrt{5}t + C_2 \sin \sqrt{5}t - \frac{7}{5}$

$$\textcircled{1} (D+1)x + (D-1)y = 2$$

$$\textcircled{2} 3x + (D+2)y = -1$$

$$(D+2)(D+1)x + (D-1)(D+2)y = 4$$

$$\text{elim } y: -3(D-1)x - (D-1)(D+2)y = -1$$

$$(D^2 + 3D + 2 - 3D + 3)x = 3 \Rightarrow (D^2 + 5)x = 3$$

$$m^2 + 5 = 0 \Rightarrow m = \pm i\sqrt{5}$$

$$x_c = C_3 \cos \sqrt{5}t + C_4 \sin \sqrt{5}t$$

oddly positioned recap:

Path 1: elim x . solve for y . — c_1, c_2

Sub $y \rightarrow \textcircled{2}$ solve for x ✓

↳ we inherit c_1, c_2 from y

Path 2: elim x . solve for y .

↳ c_1, c_2

Back to orig. elim y . solve for x .
↳ c_3, c_4

Sub $x, y \rightarrow \textcircled{2}$ to reconcile

coeffs. $\rightarrow c_3, c_4$ in terms
of c_1, c_2 .

too many constants, because the highest-order DE we worked with was 2nd-order

Nonhomog w/ $g(x) = 3$: $x_p = A \rightarrow x_p = \frac{3}{5}$

$$x(t) = C_3 \cos \sqrt{5} t + C_4 \sin \sqrt{5} t + \frac{3}{5}$$

$$y(t) = C_1 \cos \sqrt{5} t + C_2 \sin \sqrt{5} t - \frac{7}{5}$$

② $3C_3 \cos \sqrt{5} t + 3C_4 \sin \sqrt{5} t + \frac{9}{5} - C_1 \sqrt{5} \sin \sqrt{5} t + C_2 \sqrt{5} \cos \sqrt{5} t + 2C_1 \cos \sqrt{5} t + 2C_2 \sin \sqrt{5} t - \frac{14}{5} = -1$

$$3C_3 + C_2 \sqrt{5} + 2C_1 = 0$$

$$3C_4 - C_1 \sqrt{5} + 2C_2 = 0$$

$$\hookrightarrow C_3 = -\frac{\sqrt{5}}{3} C_2 - \frac{2}{3} C_1$$

$$C_4 = \frac{\sqrt{5}}{3} C_1 - \frac{2}{3} C_2$$

Example: $\frac{dx}{dt} + \frac{dy}{dt} = e^t$
 $-\frac{d^2x}{dt^2} + \frac{dx}{dt} + x + y = 0$

$\rightarrow \begin{cases} \textcircled{1} Dx + Dy = e^t \\ \textcircled{2} (-D^2 + D + 1)x + y = 0 \end{cases}$

Apply $-D$ to ②

$$(D^3 - D^2 - D)x - Dy = 0$$

$$Dx + Dy = e^t$$

$$(D^3 - D^2)x = e^t \rightarrow m^2(m-1) = 0$$

$m = 0$ (mult 2), 1 $x_c(t) = C_1 + C_2 t + C_3 e^t$

~~$x_p = A e^t$~~ $x_p = A t e^t$

$$x_p' = A(1+t)e^t, x_p'' = A(2+t)e^t$$

$$x_p''' = A(3+t)e^t$$

$$A(3+t) - A(2+t) = 1 \Rightarrow A = 1$$

$m = 0 \Rightarrow x_1 = e^0, x_2 = t e^0, x_3 = e^t$ $m = 1$

$$X(t) = c_1 + c_2 t + c_3 e^t + t e^t$$

$$\textcircled{2}: y = x'' - x' - x$$

$$x' = c_2 + c_3 e^t + (1+t)e^t$$

$$x'' = c_3 e^t + (2+t)e^t$$

$$y = \cancel{c_3 e^t} + \cancel{(2+t)e^t} - \cancel{c_2} - \cancel{c_3 e^t} - \cancel{(1+t)e^t} - c_1 - c_2 t - c_3 e^t - t e^t$$

$$\Rightarrow X(t) = c_1 + c_2 t + c_3 e^t + t e^t$$

$$y(t) = -c_1 - c_2 - c_2 t + (1 - c_3) e^t - t e^t$$

3x3 → 2x2

$$\textcircled{1} \quad Dx \quad + Dz = e^t$$

$$\text{Example } \textcircled{2} \quad (D-1)x + Dy + Dz = 0$$

$$\textcircled{3} \quad x + 2y + Dz = e^t$$

$$2 \cdot \textcircled{2} \quad (2D-2)x + 2Dy + 2Dz = 0$$

$$-D \textcircled{3} \quad -Dx - 2Dy - D^2 z = -e^t$$

$$\textcircled{4} \quad (D-2)x + (-D^2 + 2D)z = -e^t$$

$$(D-2) \textcircled{1} \quad D(D-2)x + (D^2 - 2D)z = -e^t$$

$$-D \textcircled{4} \quad -D(D-2)x + (D^3 - 2D^2)z = e^t$$

$$\textcircled{5} \quad (D^3 - D^2 - 2D)z = 0$$

$$m(m^2 - m - 2) = 0 \Rightarrow m = 0, 2, -1$$

$$z(t) = c_1 + c_2 e^{2t} + c_3 e^{-t} \rightarrow \textcircled{1}$$

$$\textcircled{1} \quad Dx + Dz = e^t$$

$$\textcircled{4} \quad (D-2)x + (-D^2 + 2D)z = -e^t$$

$$Dx + Dz = e^t \Rightarrow Dx = e^t - Dz$$

$$Dx = e^t - zc_2 e^{2t} + c_3 e^{-t}$$

$$x(t) = e^t - c_2 e^{2t} - c_3 e^{-t} + c_4$$

$$z(t) = c_1 + c_2 e^{2t} + c_3 e^{-t}$$

$$\rightarrow Dz = 2c_2 e^{2t} - c_3 e^{-t}$$

$$\rightarrow Dx + Dz = e^t$$

$$(D-1)x + Dy + Dz = 0$$

$$\rightarrow x + 2y + Dz = e^t$$

$$y = \frac{1}{2} e^t - \frac{1}{2} x - \frac{1}{2} Dz$$

$$y = \frac{1}{2} e^t$$

$$- \frac{1}{2} e^t + \frac{1}{2} c_2 e^{2t} + \frac{1}{2} c_3 e^{-t} - \frac{1}{2} c_4$$

$$- c_2 e^{2t} + \frac{1}{2} c_3 e^{-t}$$

$$y = -\frac{1}{2} c_2 e^{2t} + c_3 e^{-t} - \frac{1}{2} c_4$$

To reconcile c_4 : $(0-1)x + Dy + Dz = 0$

$$e^t - zc_2 e^{2t} + c_3 e^{-t} - e^t + c_2 e^{2t} + c_3 e^{-t} - c_4$$

$$-c_2 e^{2t} - c_3 e^{-t} + 2c_2 e^{2t} - c_3 e^{-t} = 0 \Rightarrow c_4 = 0$$

$$x(t) = e^t - c_2 e^{2t} - c_3 e^{-t}$$

$$y(t) = -\frac{1}{2} c_2 e^{2t} + c_3 e^{-t}$$

$$z(t) = c_1 + c_2 e^{2t} + c_3 e^{-t}$$

